

Singular Classical Solutions and Tree Multiparticle Cross Sections in Scalar Theories¹

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Abstract

We consider the features of multiparticle tree cross sections in scalar theories in the framework of a semiclassical approach. These cross sections at large multiplicities have exponential form, and the properties of the exponent in different regimes are discussed.

1 Introduction

Considerable interest has been attracted in recent years to the issue of multiparticle production both in perturbative and non-perturbative regimes in weakly coupled scalar field theory (for a review, see [1]). This problem has been initiated by the qualitative observation [2, 3] that in the ordinary theory with the action

$$S = \int d^{d+1}x \left(\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4 \right) \quad (1)$$

(hereafter we set the mass of the boson equal to unity) cross sections of the processes of creation of a large number of bosons by few initial ones exhibit factorial dependence on the multiplicity of the final state. The reason is that the number of tree graphs contributing to the amplitude to produce n particles grows as $n!$. At $n \sim 1/\lambda$ this factor is sufficient to compensate the suppression due to the small coupling constant, and the tree level multiparticle cross sections become large.

Now there is a lot of perturbative results (see, for instance, [4]) which confirm the factorial growth of the tree level amplitudes, and, what is more important, exhibit the exponential behavior of multiparticle cross sections

$$\sigma(E, n) \propto \exp \left(\frac{1}{\lambda} F(\lambda n, \varepsilon) \right) \quad (2)$$

where $\varepsilon = (E - n)/n$, and E is the energy of initial state. Though practically all these results have been obtained in the perturbative regime $\lambda n \ll 1$, $\varepsilon \ll 1$, one expects that the exponential behavior survives in the regime $\lambda \rightarrow 0$, with λn , ε being fixed. Moreover, there are strong indications [5] that the exponent $F(\lambda n, \varepsilon)$ is universal, i.e. independent of the few-particle initial state.

The exponential form of $\sigma(E, n)$ implies that for the calculation of the function $F(\lambda n, \varepsilon)$ there may exist a semiclassical method like the Landau technique [6] for calculating matrix elements in quantum mechanics. In fact, some approaches aiming to generalize the Landau method have been proposed recently [7, 8, 9].

Let us concentrate on the technique proposed in [7]. It is based on singular solutions of some classical boundary value problem. As the result, the function $F(\lambda n, \varepsilon)$ obtained by perturbative calculations [4] has been reproduced by means of this way. In the $(d+1)$ -space-time it looks like

$$F(\lambda n, \varepsilon) = \lambda n \ln \frac{\lambda n}{16} - \lambda n + \frac{d\lambda n}{2} \left(\ln \frac{\varepsilon}{\pi d} + 1 \right) + \frac{3d-26}{12} \lambda n \varepsilon + B \lambda^2 n^2 + O(\lambda^3 n^3) + O(\lambda^2 n^2 \varepsilon) + O(\lambda n \varepsilon^2) \quad (3)$$

where B is some known numerical constant. Moreover, certain new properties of F have been obtained in different regimes. The given method is very promising since it allows one to study $F(\lambda n, \varepsilon)$ by the perturbative expansion or, in any case, to put this issue in the computer.

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In this paper we study the behavior of $F(\lambda n, \varepsilon)$ in the regime $\lambda n \ll 1$, with ε being fixed, i.e. in fact we will investigate only the energy dependence of the tree level cross sections. We present first few terms in the low energy expansion of F in powers of ε (up to $\lambda n \varepsilon^2$). A lower bound on the function F at any energies is obtained by numerical calculations. This bound seems to be larger than the estimate found by Voloshin [10] from the analysis of the Feynman diagrams and coincides with the asymptotics at the high energies [9]. We demonstrate also that in fact the function F lies above this bound and does not coincide with it.

The paper is organized as follows. In Sect.2 we present a calculation of the boundary value problem for the tree level cross sections suggested in [11] leading to the result which coincides with one obtained from the more general approach [7]. In Sect.3 we find the $O(\varepsilon^2)$ -correction. Sect.4 is devoted to the proof that, at least in high energy limit, spherically symmetric solutions do not provide the true value of F . In Sect.5 some numerical results are presented. In Sect.6 we present our conclusions. Several technical details are collected in Appendix.

2 General formalism

Now with the aim to introduce our notations and to expose some points of the calculation we will briefly discuss the method suggested in [11]. We will examine the cross section of the decay of one virtual particle at rest with the energy E into n real particles in the model (1). Let us start from the coherent state representation (see [12] for details) for the matrix element $\langle \beta | S \varphi | 0 \rangle$. In general, it has the form

$$\langle \beta | S \varphi | 0 \rangle = \lim_{T_f \rightarrow \infty} \int D\varphi D\varphi_f \varphi(E, \mathbf{k}) \exp\{iS + B_f(\beta_k^*, \varphi_f)\}$$

where

$$\begin{aligned} \varphi(E, \mathbf{k}) &= \int dt d^d x \varphi(t, \mathbf{x}) e^{-iEt + i\mathbf{k}\mathbf{x}}, \quad \varphi_k(t) = \int \frac{d^d x}{(2\pi)^{d/2}} \varphi(t, \mathbf{x}) e^{-i\mathbf{k}\mathbf{x}}, \quad \varphi_f(\mathbf{k}) = \varphi_k(T_f) \\ B_f(\beta_k^*, \varphi_f) &= -\frac{1}{2} \int d^d k \beta_k^* \beta_{-k}^* e^{2i\omega T_f} - \frac{1}{2} \int d^d k \omega \varphi_f(\mathbf{k}) \varphi_f(-\mathbf{k}) + \int d^d k \sqrt{2\omega} e^{i\omega T_f} \beta_k^* \varphi_f(-\mathbf{k}), \quad \omega = \sqrt{1 + \mathbf{k}^2} \end{aligned}$$

To calculate this matrix element at the tree level one should extremize $(iS + B_f)$ over φ and φ_f . This extremization yields the classical field equation with the boundary conditions

$$\partial_\mu^2 \varphi + \varphi + \lambda \varphi^3 = 0, \quad \varphi_k(t \rightarrow +\infty) = \frac{\beta_k^*}{\sqrt{2\omega}} e^{i\omega t}, \quad \varphi_k(t \rightarrow -\infty) \sim a_k e^{i\omega t} \quad (4)$$

The solution to this equation, $\varphi_c(\beta^*, x)$ has only positive frequency parts due to the boundary conditions, and hence the action S and the boundary term B_f are reduced to zero. Thus, the tree level matrix element is

$$F(\beta^*) \equiv \langle \{\beta\} | S \varphi | 0 \rangle_{\text{tree}} = \int dt d^d x \varphi_c(\beta^*, t, \mathbf{x}) e^{-iEt + i\mathbf{k}\mathbf{x}}$$

and the amplitude $1 \rightarrow n$ is given by

$$A_n = \frac{\partial^n}{\partial \{\beta^*\}^n} F(\beta^*) \Big|_{\{\beta^*\}=0}$$

This result coincides with one obtained in [13] from LSZ-reduction formula.

According to the coherent state formalism the n -particle cross section can be easily found,

$$\sigma_{1 \rightarrow n}^{\text{tree}} = \oint \frac{d\xi}{\xi} \int D\beta D\beta^* dx dx' \varphi_c(\beta^*, x) \varphi_c^*(\beta, x') \exp \left\{ -iE(t - t') + i\mathbf{k}(\mathbf{x} - \mathbf{x}') - \frac{1}{\xi} \int d^d k \beta^* \beta - n \ln \xi \right\} \quad (5)$$

where the integral over ξ corresponds to the n -th derivative due to Cauchy theorem. In order to estimate the integral (5) the saddle point technique can be applied. To perform it correctly, however, one should look more carefully at φ_c and get rid of zero modes in φ_c corresponding to time shifts. One writes

$$\beta_k^* = b_k^* e^{i\omega\eta} \Rightarrow \varphi_c(\beta^*, t) = \varphi(b^*, t + \eta)$$

where η is the collective coordinate, and b_k are new integration variables which obey a constraint to be specified below. (Of course, there are zero modes corresponding to the space shifts, but we do not write them explicitly since taking them into account leads only to spacial momentum conservation). So, Eq.(5) becomes

$$\sigma_{1 \rightarrow n}^{\text{tree}} = \oint \frac{d\xi}{\xi} \int DbDb^* d\eta d\eta' \varphi(b^*, E, \mathbf{k}) \varphi^*(b, E, \mathbf{k}) \exp \left\{ -iE(\eta - \eta') - \frac{1}{\xi} \int d^d k b^* b e^{i\omega(\eta' - \eta)} - n \ln \xi \right\}$$

Now, if one imposes a constraint that φ does not contain exponents and, so, may be considered as a pre-exponential factor, then this integral can be calculated by making use of the saddle point technique. The exact form of this constraint, which may be viewed as a constraint on b_k^* , will be determined in what follows. Therefore, the tree cross section is

$$\sigma(E, n)^{\text{tree}} \propto e^{W_{\text{extr}}} \quad (6)$$

where W_{extr} is the saddle point value of the functional

$$W = ET - n\theta - \int d^d k b_k^* b_k e^{\omega T - \theta} \quad (7)$$

with respect to $T = i(\eta' - \eta)$, $\theta = \ln \xi$, b_k , and b_k^* under an additional constraint which is yet to be defined.

In general, one expects that these classical solutions, being considered in Euclidean space-time, have singularities on some surface $\tau = \tau(\mathbf{x})$. The leading behavior of the integral

$$\varphi(E, \mathbf{k}) = \int d^{d+1} x \varphi(x) e^{-iEt + i\mathbf{k}\mathbf{x}}$$

is determined by the nearest to the origin singularity of $\varphi(x)$ in the region $-\tau = \text{Im } t < 0$. If this singularity is located at $\tau = \tau^*$, then

$$\varphi(E, \mathbf{k}) \propto e^{-E\tau^*}$$

So, the constraint that φ may be considered as a pre-exponential factor requires $\tau^* = 0$. Since we are working in the center-of-mass frame, the result should depend on $|\mathbf{k}|$, but not on its direction. So, in \mathbf{x} -space we require $O(3)$ spherical symmetry and finally the constraint is that $\varphi(x)$ has the singularity at $\tau = 0$, $\mathbf{x} = 0$.

Therefore, the problem of finding the cross section at any value of E and n can be formulated in Euclidean space-time and consists of the following steps,

- One selects from all solutions $\varphi(\tau, \mathbf{x})$ to the Euclidean field equation, $O(3)$ -symmetrical ones which are singular at $\tau = \mathbf{x} = 0$ and have the following asymptotics at $\tau \rightarrow \infty$,

$$\int \frac{d^d x}{(2\pi)^{d/2}} \varphi(\tau, \mathbf{x}) e^{-i\mathbf{k}\mathbf{x}} = \frac{b_k^*}{\sqrt{2\omega}} e^{-\omega\tau} \quad (8)$$

From them one finds Fourier components b_k and then determines W from Eq.(7).

- One should extremize W over all b_k , b_k^* (that is over different singular surfaces), T , and θ . The tree level cross section is then given by the formula (6).

Now let us see that the extremum of W at the fixed energy ε and n is in fact its maximum. Indeed, let us consider the functional

$$\int d^d k b_k b_k^* e^{\omega T}$$

This functional is bounded below, so it has a minimum at some value of b_k (at this point W as a function of b_k has its maximum),

$$\int d^d k b_k b_k^* e^{\omega T} \Big|_{\min} = C(T) > \text{const}$$

If one fixes some particular surface of singularities Σ , than the obtained value of the functional

$$\int d^d k b_k b_k^* e^{\omega T} \Big|_{\Sigma_{\text{fixed}}} = C_{\Sigma}(T) \geq C(T)$$

for all values of T . Taking a saddle point value of θ , one gets $\theta = -\ln n + \ln C(T)$ and

$$W(T) = n \ln n - n + ET - n \ln C(T) \quad \text{has an extremum at } T_1,$$

$$W(T)_{\Sigma} = n \ln n - n + ET - n \ln C_{\Sigma}(T) \quad \text{has an extremum at } T_2$$

Comparing $W(T_1)$ and $W(T_2)$ the following chains can be obtained

$$W(T_1) \geq W(T_2) \geq W_{\Sigma}(T_2) \quad \text{if } T_1 \text{ provides a maximum of } W$$

$$W_{\Sigma}(T_2) \leq W_{\Sigma}(T_1) \leq W(T_1) \quad \text{if } T_2 \text{ provides a minimum of } W_{\Sigma}$$

Therefore, if one fixes some class of surfaces of singularities and energy ε or, in other words, finds some particular singular solution, than one can get a lower bound for $W(E, n)$.

Due to the nonlinearity of the field equations it is not possible to obtain an analytical solution to the boundary value problem formulated above. One can find, however, analytical solutions in several regimes.

3 Low energy expansion of $F(\lambda n, \varepsilon)$

In this section we calculate $F(\lambda n, \varepsilon)$ to the accuracy of $O(\varepsilon^3)$. The typical momentum of final particles at small ε is much smaller than the mass, so one can expect that the solution to the field equation φ_c is a slowly varying function of x . The x -independent solution having singularity at $\tau = 0$ and decaying at $\tau \rightarrow \infty$ can be found exactly [13],

$$\varphi_0(\tau) = \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh \tau}$$

It corresponds, however, to the case of all final particles being at rest. Let us modify this solution by imposing by hand dependence on \mathbf{x} . One writes

$$\varphi_0(\tau, \mathbf{x}) = \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh(\tau - \tau_0(\mathbf{x}))}$$

This function has the surface of singularities $\tau = \tau_0(\mathbf{x})$. In the end we will extremize the functional W , Eq.(7), with respect to $\tau_0(\mathbf{x})$. According to the general formalism described above, $\tau_0(0)$ must vanish, and due to spherical symmetry $\tau_0(\mathbf{x})$ depends only on $|\mathbf{x}|$. Starting from this function (which satisfies the field equation to the accuracy of $O((\partial_{\mathbf{x}}\tau_0)^2)$) one can expand φ in the following way,

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \dots \quad (9)$$

where φ_1, φ_2 are of order of $(\partial_{\mathbf{x}}\tau_0)^2, (\partial_{\mathbf{x}}\tau_0)^4$ respectively. The explicit form of φ_1, φ_2 can be found in Appendix.

In order to obtain b_k^* corresponding to the solution, let us find the asymptotics of φ at $\tau \rightarrow \infty$ (see Eq.(41), $z = \tau - \tau_0(\mathbf{x})$),

$$\varphi^{\text{as}} = \sqrt{\frac{2}{\lambda}} e^{-z} \left\{ 2 - \frac{5}{6} \partial^2 \tau_0 + (\partial^2 \tau_0 + (\partial \tau_0)^2) z + \partial^4 \tau_0 \left(\frac{z^2}{4} - \frac{z}{6} - \frac{1}{8} \right) + \partial^2 (\partial \tau_0)^2 \left(\frac{z^2}{4} + \frac{z}{4} - \frac{3 + \pi^2}{24} \right) \right\}$$

$$\begin{aligned}
& +\partial\tau_0\partial^3\tau_0\left(\frac{z^2}{2}-\frac{4z}{3}+\frac{3+\pi^2}{12}\right)+\partial\tau_0\partial(\partial\tau_0)^2\left(\frac{z^2}{2}-\frac{z}{2}+\frac{5}{12}\right)+(\partial^2\tau_0)^2\left(\frac{z^2}{4}-\frac{2z}{3}+\frac{\pi^2-7}{12}\right) \\
& +(\partial\tau_0)^2\partial^2\tau_0\left(\frac{z^2}{2}-\frac{17z}{12}+\frac{5}{12}\right)+(\partial\tau_0)^4\left(\frac{z^2}{4}-\frac{3z}{4}\right)\}
\end{aligned} \tag{10}$$

This expression is of course the solution to the accuracy of $O((\partial\tau_0)^6)$ to the linearized field equation which has the following general solution,

$$\varphi(x, \tau) = \int \frac{d^d k}{(2\pi)^{d/2}} \frac{b_k^*}{\sqrt{2\omega}} e^{i\mathbf{k}\mathbf{x} - \omega\tau}, \quad \omega = \sqrt{1+k^2} \tag{11}$$

Let us introduce the notation

$$b(x) = \sqrt{\frac{\lambda}{2}} \int \frac{d^d k}{(2\pi)^{d/2}} \frac{b_k^*}{\sqrt{2\omega}} e^{i\mathbf{k}\mathbf{x}} \tag{12}$$

Thus, one gets, for any function of the form (11), the following relation,

$$\varphi(\mathbf{x}, \tau) = \int \frac{d^d k}{(2\pi)^{d/2}} \frac{b_k^*}{\sqrt{2\omega}} e^{i\mathbf{k}\mathbf{x} - \tau} \left(1 - \frac{\tau}{2}k^2 + \frac{\tau + \tau^2}{8}k^4\right) = e^{-\tau} \sqrt{\frac{2}{\lambda}} \left(b + \frac{\tau}{2}\partial^2 b + \frac{\tau + \tau^2}{8}\partial^4 b\right) \tag{13}$$

Assuming that

$$b = 2e^{\tau_0} + b_1 + b_2 \tag{14}$$

where $b_1 \sim (\partial\tau_0)^2$, $b_2 \sim (\partial\tau_0)^4$, one can find from (13) that to the accuracy of $O((\partial\tau_0)^6)$,

$$\varphi = \sqrt{\frac{2}{\lambda}} e^{-\tau} \left(2e^{\tau_0} + \tau\partial^2 e^{\tau_0} + b_1 + \frac{\tau + \tau^2}{4}\partial^4 e^{\tau_0} + \frac{\tau}{2}\partial^2 b_1 + b_2\right) \tag{15}$$

Comparing the terms of the same order in $\partial\tau_0$ in Eqs.(15) and (10) one finds for $b_1(0)$ and $b_2(0)$,

$$b_1(0) = -\frac{5}{6}\partial^2\tau_0, \quad b_2(0) = -\frac{\partial^4\tau_0}{8} - \frac{3+\pi^2}{12}(\partial_{ij}^2\tau_0)^2 + \frac{\pi^2-7}{12}(\partial^2\tau_0)^2 \tag{16}$$

To get these expressions we took into account that $\tau_0(0) = 0$ and due to the $O(3)$ spherical symmetry, $\partial_i\tau_0(x)|_{x=0} = 0$. From Eqs.(16) and (14) one finds the following system of the equations for b and its derivatives at $\mathbf{x} = 0$,

$$\begin{aligned}
b &= 2 - \frac{5}{6}\partial^2\tau_0 - \frac{1}{8}\partial^4\tau_0 - \frac{3+\pi^2}{12}(\partial_{ij}^2\tau_0)^2 + \frac{\pi^2-7}{2}(\partial^2\tau_0)^2; \\
\partial_{ij}^2 b &= 2\partial_{ij}^2\tau_0 - \frac{5}{6}\partial_{ij}^2\tau_0\partial^2\tau_0 - \frac{11}{6}\partial_{ij}^2\partial^2\tau_0, \quad \partial^4 b = 2\partial^2\tau_0 + 4(\partial_{ij}^2\tau_0)^2 + 2(\partial^2\tau_0)^2
\end{aligned} \tag{17}$$

From this system one can easily find all the derivatives of τ_0 with respect to \mathbf{x} at $\mathbf{x} = 0$ via the derivatives of b ,

$$\partial^2\tau_0 = \frac{1}{2}\partial^2 b + \frac{5}{24}\partial^4 b - \frac{5}{24}(\partial_{ij}^2 b)^2 + \frac{1}{8}(\partial^2 b)^2, \quad (\partial_{ij}^2\tau_0)^2 = \frac{1}{4}(\partial_{ij}^2 b)^2, \quad \partial^4\tau_0 = \frac{1}{2}\partial^4 b - \frac{1}{2}(\partial_{ij}^2 b)^2 - \frac{1}{4}(\partial^2 b)^2 \tag{18}$$

Substituting these solutions into the first equation of (17) one gets the constraint on b ,

$$2 = b + \frac{5}{12}\partial^2 b + \frac{17}{72}\partial^4 b + \frac{3\pi^2-25}{144}(\partial_{ij}^2 b)^2 + \frac{21-2\pi^2}{96}(\partial^2 b)^2 \tag{19}$$

From (12) and (19) one obtains

$$\sqrt{\frac{16}{\lambda}} = \sqrt{\frac{\lambda}{4}} \frac{3\pi^2-25}{144} \left(\int \frac{d^d k}{(2\pi)^{d/2}} \frac{k_i k_j b_k^*}{\sqrt{\omega}} \right)^2 + \sqrt{\frac{\lambda}{4}} \frac{21-2\pi^2}{96} \left(\int \frac{d^d k}{(2\pi)^{d/2}} \frac{k^2 b_k^*}{\sqrt{\omega}} \right)^2$$

$$+ \int \frac{d^d k}{(2\pi)^{d/2}} \frac{b_k^*}{\sqrt{\omega}} \left(1 - \frac{5}{12} k^2 + \frac{17}{72} k^4 \right) \equiv \mathcal{F}(b_k^*) \quad (20)$$

Let us now to extremize (7) taking into account the constraint (20). Making use of the Lagrange multiplier method, we rewrite (7) in the following form

$$W = ET - n\theta - \int d^d k b_k b_k^* e^{\omega T - \theta} + A \left(\mathcal{F}(b_k^*) - \sqrt{\frac{16}{\lambda}} \right) \quad (21)$$

where the constant A should be determined. Now W can be varied with respect to b_k^* without any restrictions. Performing this procedure one gets

$$b_k = \frac{A e^{-\omega T + \theta}}{(2\pi)^{d/2} \sqrt{\omega}} \left(1 - \frac{5}{12} k^2 + \frac{17}{72} k^4 + 2\sqrt{\frac{\lambda}{4}} \left(\frac{3\pi^2 - 25}{144} k_i k_j C_{ij} + \frac{21 - 2\pi^2}{96} k^2 \text{Tr} C_{ij} \right) \right) \quad (22)$$

where

$$C_{ij} = \int \frac{d^d k}{(2\pi)^{d/2}} \frac{k_i k_j}{\sqrt{\omega}} b_k^* \quad (23)$$

Assuming $b_k = b_k^*$, from (22) and (23) one obtains the equation for C_{ij} ,

$$C_{ij} = A \int \frac{d^d k}{(2\pi)^d} \frac{k_i k_j}{\omega} e^{-\omega T + \theta} \left(1 - \frac{5}{12} k^2 + \frac{17}{72} k^4 + 2\sqrt{\frac{\lambda}{4}} \left(\frac{3\pi^2 - 25}{144} k_n k_m C_{nm} + \frac{21 - 2\pi^2}{96} k^2 \text{Tr} C_{nm} \right) \right) \quad (24)$$

Let us now concentrate on the equation (24). First of all we consider the case $i \neq j$. Due to the fixed parity of the integrand,

$$C_{ij} = \sqrt{\frac{\lambda}{4}} \frac{A}{(2\pi)^d} \frac{3\pi^2 - 25}{36} C_{ij} \int \frac{d^d k}{\omega} (k_i)^2 (k_j)^2 e^{-\omega T + \theta}$$

(no summations over repeated indices). From the last equation one concludes that $C_{ij} = \delta_{ij} \text{Tr} C_{nm} / d \equiv \delta_{ij} C / d$. Together with Eq.(24), this leads to

$$C = \frac{A}{(2\pi T)^{d/2}} e^{\theta - T} \left(\frac{d}{T} + O\left(\frac{1}{T^2}\right) \right) \quad (25)$$

T is assumed to be large at small ε since, in fact, $T \sim W_{\text{extr}}/E$. The constant A can then be found from the constraint (20) by making use of Eqs.(22) and (25):

$$A = \sqrt{\frac{16}{\lambda}} \frac{e^{T - \theta}}{(2\pi T)^{d/2}} \left(1 - \frac{d(3d - 26)}{24T} + \frac{d}{T^2} \left(\frac{9d^3 - 156d^2 + 884d - 1632}{1152} - 3d\delta \right) \right) \quad (26)$$

where

$$\delta \equiv \frac{3\pi^2 - 25}{72d} + \frac{21 - 2\pi^2}{48}$$

So, one finally obtains the expression for b_k ,

$$b_k = \sqrt{\frac{16}{\lambda}} \frac{e^{T - \omega T} T^{d/2}}{\omega} \left(1 - \frac{d(3d - 26)}{24T} + \frac{d}{T^2} \left(\frac{9d^3 - 156d^2 + 884d - 1632}{1152} - 3d\delta \right) \right) \times \left(1 + \left(\frac{2d\delta}{T} - \frac{5}{12} \right) k^2 + \frac{17}{72} k^4 \right) \quad (27)$$

The variation of the functional (21) with respect to θ yields to

$$n = \frac{A^2 e^{\theta - T}}{(2\pi T)^{d/2}} \left(1 + \frac{d(3d - 26)}{24T} + \frac{d}{T^2} \left(\frac{(d + 2)(3d^2 - 58d + 272)}{384} + 4d\delta \right) \right) \quad (28)$$

From (28) θ can be found as a function of T and n . The variation of (21) with respect to T gives the equation for T ,

$$\varepsilon = \frac{d}{2T} \left(1 + \frac{3d-26}{12T} - \frac{13d-102}{18T^2} + \frac{8d\delta}{T^2} \right) \Rightarrow T = \frac{d}{2\varepsilon} + \frac{3d-26}{12} + \varepsilon \left(16\delta - \frac{9d^2-52d-140}{72d} \right) \quad (29)$$

T is seen to be indeed large at small ε .

Collecting (26), (28), and (29) one finally gets for the function W

$$W = n \left(\ln \frac{\lambda n}{16} - 1 + \frac{d}{2} \left(\ln \frac{\varepsilon}{d\pi} + 1 \right) + \varepsilon \frac{(3d-26)}{12} - \frac{\varepsilon^2}{144d} (9d^2 - 556d + 260 + 48\pi^2(d-1)) \right) \quad (30)$$

The first three terms in this expression coincide with those obtained in [7] and the last term is an ε^2 correction to W . Particularly, at $d=3$, W has the form

$$W_{d=3} = n \left(\ln \frac{\lambda n}{16} - 1 + \frac{3}{2} \left(\ln \frac{\varepsilon}{3\pi} + 1 \right) - \varepsilon \frac{17}{12} + \frac{\varepsilon^2}{432} (1327 - 96\pi^2) \right) \quad (31)$$

Finally, the last thing which is interesting to be performed is the calculation of $\tau_0(x)$ which corresponds to the obtained b_k (27). To do it, one can find all derivatives of $\tau_0(x)$ at $x=0$ (see (18)) and after that restore $\tau_0(x)$ as the Taylor's series in powers of $|\mathbf{x}|$.

So, using the definition of b (12), expression for b_k (27), and taking into account that due to spherical symmetry $\partial_{ij}^2 = \frac{\delta_{ij}}{d} \partial^2$ one finds

$$\partial^2 b|_{x=0} = -\frac{2d}{T} - \frac{d(11d-10)}{6T^2} + O\left(\frac{1}{T^3}\right), \quad \partial^4 b|_{x=0} = \frac{2d(d+2)}{T^2} + O\left(\frac{1}{T^3}\right) \quad (32)$$

All derivatives of τ_0 can be easily found from Eqs. (18) and (32),

$$\partial^2 \tau_0|_{x=0} = -\frac{d}{T} + \frac{5}{6} \frac{d}{T^2}, \quad \partial_{ij}^2 \tau_0|_{x=0} = \delta_{ij} \left(-\frac{1}{T} + \frac{5}{6T^2} \right), \quad \partial^4 \tau_0|_{x=0} = \partial_{ijkl}^4 \tau_0|_{x=0} = \partial^2 \partial_{ij}^2 \tau_0|_{x=0} = 0$$

Thus, the singularity surface which corresponds to the cross section up to $O(\varepsilon^2)$ is a paraboloid,

$$\tau_0(x) = x^2 \left(-\frac{1}{2T} + \frac{5}{12} \frac{1}{T^2} \right) = -\frac{x^2 \varepsilon}{d} \left(1 - \varepsilon \frac{3d-16}{6d} \right) \quad (33)$$

The Eq.(33) is valid for values of \mathbf{x} being smaller than the typical inverse momentum of the final particles, $\varepsilon^{-1/2}$. It is worth noting that the term of order $(x^4 \varepsilon^2)$ vanishes. These results are useful to compare with numerical calculations (see Sect.5).

4 Lower bound for $F(\lambda n, \varepsilon)$ in ultrarelativistic regime

Let us now turn to the opposite limit, $\varepsilon \gg 1$. In this case one can neglect the mass term in the field equation and consider the massless theory. Since Euclidean massless ϕ^4 theory is conformally invariant, several solutions can be found analytically. The simplest one is $O(4)$ symmetric Lipatov's instanton [14]. The corresponding solution which is singular at $\tau = \mathbf{x} = 0$ can be easily constructed [9] by making the size of the instanton pure imaginary and shifting its center:

$$\varphi_0 = \sqrt{\frac{8}{\lambda}} \frac{\rho}{\mathbf{x}^2 + (\tau + \rho)^2 - \rho^2} \quad (34)$$

The variational procedure on T , θ and only collective coordinate ρ (clearly, no collective coordinates describing the shape of the singularity surface appears) could be made and results in (see [9])

$$W = n \ln \frac{\lambda n}{16} - \lambda n + n \ln \frac{2}{\pi^2} \quad (35)$$

However, the variation on the shape of singularity surface is still required, unless the result (35) provides only lower bound for the true exponent.

To see that these $O(4)$ symmetric calculations in fact do not provide the extremum over the shape of singularities, let us consider the small perturbations about the solution (34). Since the solution should be $O(3)$ symmetric with respect to \mathbf{x} rotations, it is sufficient to consider the approximate solutions which slightly vary the singularity surface so that the dependence of ψ , an angle between the τ axis and the radius vector, appears. The solution should have the form

$$\varphi = \frac{2\sqrt{\lambda/2}\rho + \alpha f(r)C_m^{(1)}(\cos\psi)}{r^2 - \rho^2(1 + \alpha C_m^{(1)}(\cos\psi))^2} = \varphi_0 + \sum_m \alpha_m C_m^{(1)}(\cos\psi) \tilde{\varphi}_m(r) + O(\alpha^2)$$

In this equation, $C_m^{(1)}(\cos\psi)$ are Gegenbauer polynomials, α_m are small parameters, and the linear fluctuations $\tilde{\varphi}_m$ should satisfy the following equation,

$$\left[\partial_r^2 + \frac{3}{r} \partial_r - \frac{m(m+2)}{r^2} - \frac{24\rho^2}{(r^2 - \rho^2)^2} \right] \tilde{\varphi}_m(r) = 0$$

To solve this equation, one can exploit the conformal invariance of the model and use the stereographic projection onto the sphere S^4 (it is a convenient way to solve the equations in the conformal models, we use here a slightly modified method of [15]). Skipping the details, the solution is

$$\tilde{\varphi}_m(r) = \frac{1}{\sqrt{\lambda}} \frac{r^{2-m} \rho^{m+1}}{(r^2 - \rho^2)^2} \left(1 - \frac{2(m-1)}{m+2} \frac{\rho^2}{r^2} + \frac{(m-1)m}{(m+2)(m+3)} \frac{\rho^4}{r^4} \right)$$

Calculating the asymptotics of this solution, one obtains the following expression for b_k :

$$b_k = b_k^0 + \sum_m \alpha_m \sqrt{\frac{\pi}{k\lambda}} e^{-k\rho} \left(\rho^{m+1} \frac{k^m}{m!} - 3\rho^2 k \frac{(m+1)}{(m+2)(m+3)} \right)$$

(here b_k^0 corresponds to the solution (34)). Then the exponent after the variational procedure on T and θ is modified by

$$\Delta W = \sum_m \alpha_m \frac{(m+1)}{\rho} \left(2^{-m-2} - \frac{3}{2(m+2)(m+3)} \right)$$

For $m = 0$ or 1 (which corresponds to the dilatations and translations of φ_0), $\Delta W = O(\alpha^2)$. However, for all other values of m the perturbation in the exponent is proportional to α_m , so that the functional derivative of the exponent calculated at $O(4)$ -symmetric solution with respect to collective coordinates describing the shape of the singularity surface is non-zero. So, the result (35) is indeed only lower bound on the tree cross sections in high-energy limit. One could expect that for lower energies (massive theory) this property will be conserved.

5 Lower bound of $F(\lambda n, \varepsilon)$ at arbitrary ε

So, to obtain a lower bound for $\sigma_{\text{tree}}(E, n)$ at given E and n one could consider $O(4)$ symmetric field configurations [16]. The singularity surfaces are then three-spheres in Euclidean space, and the only variational parameter is the radius of a sphere, ρ . Since the singularity surface should touch the origin, the sphere should be centered at

$$x_0 = (-\rho, \mathbf{0})$$

i.e., the $O(4)$ symmetric solution has the general form

$$\varphi = \varphi(r), \quad r = \sqrt{\mathbf{x}^2 + (\tau + \rho)^2}$$

At large r , the solution tends to the exponentially falling solution to the free field equation,

$$\varphi \sim \frac{K_1(r)}{r} \sim \frac{e^{-r}}{r^{3/2}}$$

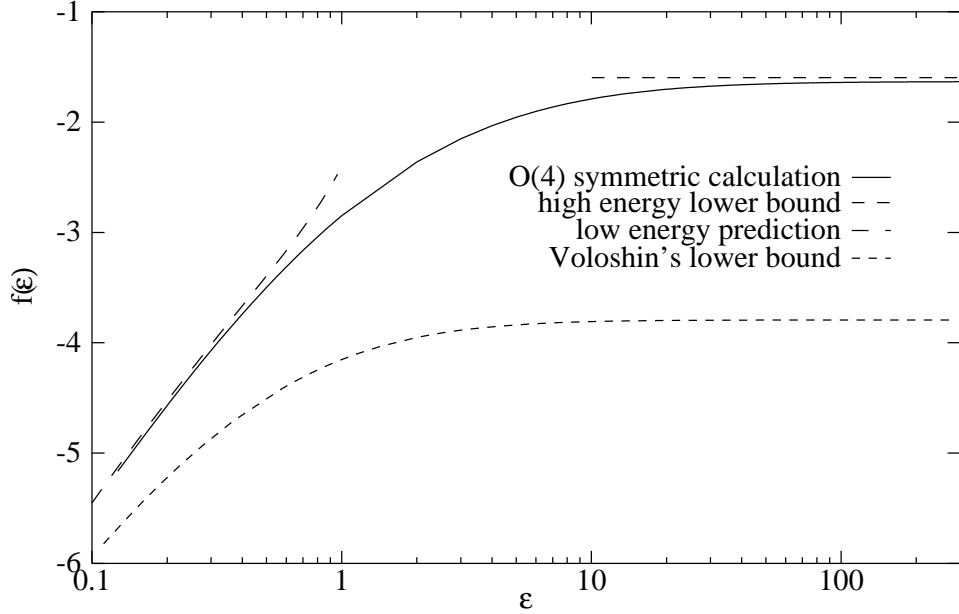


Fig. 1

i.e., at $\tau \rightarrow \infty$ one has

$$\varphi = A(\rho) \frac{\exp(-\sqrt{\mathbf{x}^2 + (\tau + \rho)^2})}{(\mathbf{x}^2 + (\tau + \rho)^2)^{3/4}}$$

where the coefficient function $A(\rho)$ is to be determined by solving the field equations under the condition that it has a singularity at $r = \rho$. From this asymptotics one finds

$$b_k = 2A(\rho) \frac{e^{-\omega\rho}}{\sqrt{2\omega}}$$

The "action" (7) is then expressed through $A(\rho)$, so W reads

$$W = ET - n\theta - 8\pi A^2(\rho) e^{-\theta} \frac{K_1(2\rho - T)}{2\rho - T} \quad (36)$$

Now we extremize Eq.(36) with respect to ρ , T and θ . This leads to the following equations which determine the saddle point values of these three parameters,

$$\frac{E}{n} = \frac{A'(\rho)}{A(\rho)} = \frac{K_2(2\rho - T)}{K_1(2\rho - T)} \quad (37)$$

$$\theta = \ln \left\{ \frac{8\pi A^2(\rho)}{n} \frac{K_1(2\rho - T)}{2\rho - T} \right\} \quad (38)$$

So, we look for the classical solutions which are singular at the spheres $r^2 = \rho^2$ and from their asymptotics obtain $A(\rho)$, then express saddle point values of ρ , T and θ through E and n (by making use of the Eqs. (37), (38)) and finally obtain the estimate for the exponent for the tree cross section (see Eq.(6))

$$W_{\text{tree}} = ET - n\theta - n$$

It is straightforward to perform this calculation numerically for all ε . The exponent for the cross section has the form

$$F(\lambda n, \varepsilon) = \lambda n \ln \frac{\lambda n}{16} - \lambda n + \lambda n f(\varepsilon) + O(\lambda^2 n^2)$$

with the function $f(\varepsilon)$ plotted in Fig.1. At low energies our result matches the perturbative results, Eq.(31). The fact that our variational approach leads to the exact results for W_{tree} at small ε can be understood as follows. At small ε , the curvature of the singularity surface is always large, and only this curvature is relevant for the evaluation of $f(\varepsilon)$ [7]. In other words, the surface of singularities has the form (see Eq.(33)),

$$\tau(\mathbf{x}) = a\mathbf{x}^2 + O(\mathbf{x}^4) + \dots$$

and only the leading term is important at small ε . Clearly, this leading term can be reproduced exactly in our $O(4)$ symmetric ansatz, and our result is exact at small ε . At very high energies (small values of ρ) our field configuration tends to the $O(4)$ symmetric solution of the massless equations, Eq.(34). So, at $\varepsilon \gg 1$ it approaches the lower bound derived from this solution. The alternative lower bound on $f(\varepsilon)$ can be easily read out from ref. [10] and it is also shown in Fig.1. This bound has been obtained by direct analysis of diagrams. As one can see from Fig.1, our bound is stronger than that of ref. [10].

6 Conclusions

To summarize, the tree multiparticle cross sections can be calculated using a semiclassical method based on singular solutions to classical field equations. At low kinetic energies ε of particles in the final state the corresponding solutions can be obtained analytically and the leading exponent in the cross section can be found at least up to $O(\varepsilon^2)$. Instead of proceeding to higher orders in ε , one can solve the field equations numerically for different surfaces of singularity and after that apply the variational procedure at arbitrary ε . For the simplest form of the singularity, the spherically symmetric one, the variational procedure over collective coordinates is incomplete and provides a lower bound on the tree cross section (the correct result at tree level should correspond to the true maximum of the exponent). To improve this estimate, one should consider more general field configurations. Beyond the tree level, the semiclassical Landau-like methods are still applicable, but the variational procedure is more complicated and corresponding solutions are known only in the simplest case of zero energies.

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Appendix

The first two terms in the expansion (9) have been obtained in [7],

$$\varphi = \varphi_0 + \varphi_1 = \sqrt{\frac{2}{\lambda}} \left\{ \frac{1}{\sinh z} + \frac{1}{2} \left(\partial^2 \tau_0 \left(\frac{\cosh z - \sinh z}{3} - \frac{1}{\sinh z} + z \frac{\cosh z}{\sinh^2 z} \right) + (\partial \tau_0)^2 z \frac{\cosh z}{\sinh^2 z} \right) \right\} \quad (39)$$

($z = \tau - \tau_0(\mathbf{x})$). The function φ_1 satisfies the following equation,

$$\partial_\tau^2 \varphi_1 - \varphi_1 - 3\lambda \varphi_0^2 \varphi_1 = -\partial_i^2 \varphi_0 = -\partial_\tau^2 \varphi_0 (\partial_i \tau_0)^2 + \partial_\tau \varphi_0 \partial_i^2 \tau_0 \quad (40)$$

The solution (39) can be found with the help of the Green function,

$$\varphi = \varphi_0 + \int^z d\sigma K(z, \sigma) (\partial_\sigma \varphi_0 \partial^2 \tau_0 - \partial_\sigma^2 \varphi_0 (\partial \tau_0)^2) + C_1(x) f_1(z) + C_2(x) f_2(z)$$

where

$$f_1(\tau) = -\partial_\tau \varphi_0 = \sqrt{\frac{2}{\lambda}} \frac{\cosh(z)}{\sinh^2(z)}, \quad f_2(\tau) = \sqrt{\frac{2}{\lambda}} \frac{\cosh(z)}{\sinh^2(z)} \left(\frac{\sinh 2(z)}{4} - \frac{3(\tau - \tau_0)}{2} + \tanh(z) \right)$$

are the solutions of the homogeneous Eq.(40) and

$$K(z, \sigma) = \frac{\lambda}{2} (f_2(z) f_1(\sigma) - f_2(\sigma) f_1(z))$$

is the Green function. The functions $C_1(x)$ and $C_2(x)$ can be found from the conditions $\varphi(\tau \rightarrow \infty, x) \rightarrow 0$ and $\varphi(0, 0) \rightarrow \infty$.

The next correction (we expect that it is of order of $(\partial^4 \tau_0)$) can be obtained in the same way from the following equation,

$$\partial_\tau^2 \varphi_2 - \varphi_2 - 3\lambda \varphi_0^2 \varphi_2 = -\partial_i^2 \varphi_1 + 3\lambda \varphi_1^2 \varphi_0$$

and has the following form,

$$\begin{aligned} \varphi_2 = \sqrt{\frac{2}{\lambda}} \Big\{ & -\partial_i^2 \partial_j^2 \tau_0 \left(\frac{1}{8} \frac{1}{\sinh z} - \frac{1}{8} (\cosh z - \sinh z) - \frac{1}{12} z (\cosh z - \sinh z) + \frac{1}{4} \frac{z}{\sinh z} - \frac{1}{8} z \frac{\cosh z}{\sinh^2 z} \right. \\ & - \frac{1}{8} z^2 \frac{\cosh z}{\sinh^2 z} \Big) - \partial_j^2 (\partial_i \tau_0)^2 \left(\frac{1}{12} \frac{1}{\sinh z} - \frac{1}{3} z \frac{\cosh z}{\sinh^2 z} - \frac{1}{12} z \cosh z + I(z) \right) + 2\partial_j \partial_i^2 \tau_0 \cdot \partial_j \tau_0 \left(\frac{1}{12} \frac{1}{\sinh z} \right. \\ & - \frac{1}{3} z \frac{\cosh z}{\sinh^2 z} - \frac{1}{4} \frac{z}{\sinh z} + \frac{1}{4} z^2 \frac{\cosh z}{\sinh^2 z} - \frac{1}{12} z \sinh z + I(z) \Big) + 2\partial_j [(\partial_i \tau_0)^2] \cdot \partial_j \tau_0 \left(\frac{1}{8} \frac{1}{\sinh z} - \frac{1}{8} z \frac{\cosh z}{\sinh^2 z} \right. \\ & - \frac{1}{24} (\cosh z - \sinh z) + \frac{1}{8} z^2 \frac{\cosh z}{\sinh^2 z} \Big) + \partial_i^2 \tau_0 \partial_j^2 \tau_0 \left(\frac{1}{12} z \cosh z - \frac{1}{4} z \sinh z + \frac{1}{4} (\cosh z - \sinh z) - \frac{3}{8} \frac{1}{\sinh z} \right. \\ & - \frac{3}{8} z \frac{\cosh z}{\sinh^2 z} - \frac{1}{2} \frac{z}{\sinh z} + \frac{1}{4} \frac{z^2}{\sinh^3 z} + \frac{1}{8} \frac{z^2}{\sinh z} + \frac{1}{4} z^2 \frac{\cosh z}{\sinh^2 z} + 2I(z) \Big) + (\partial_i \tau_0)^2 \partial_j^2 \tau_0 \left(\frac{1}{12} z (\cosh z - \sinh z) \right. \\ & - \frac{1}{12} (\cosh z - \sinh z) + \frac{1}{2} \frac{z^2}{\sinh^3 z} + \frac{1}{4} \frac{z^2}{\sinh z} + \frac{1}{4} \frac{1}{\sinh z} - \frac{3}{4} z \frac{\cosh z}{\sinh^2 z} \Big) \\ & \left. - (\partial_i \tau_0)^2 (\partial_j \tau_0)^2 \left(\frac{3}{8} z \frac{\cosh z}{\sinh^2 z} - \frac{1}{4} \frac{z^2}{\sinh^3 z} - \frac{1}{8} \frac{z^2}{\sinh z} \right) \right\} \end{aligned} \quad (41)$$

where

$$I(z) = \frac{1}{12} \ln(2 \sinh z) \left(\sinh z + \frac{3}{\sinh z} - 3z \frac{\cosh z}{\sinh^2 z} \right) + \frac{1}{4} \frac{\cosh z}{\sinh^2 z} \int_0^z x \coth x dx.$$

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